Solutions to AMM 2026

American Masters in Mathematics 2026

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Contents

§1 Solution to AMM 2026/1, proposed by Serena An

Determine whether there exists a prime p and a positive integer N with the following property: given any N-digit positive integer whose decimal digits are all nonzero, it is possible to permute its digits to obtain a multiple of p.

No. Suppose p divided $\underbrace{1\cdots 11}_{N}$ and some rearrangement of $\underbrace{1\cdots 1}_{N-1}2$. Then p would also divide their difference, a power of 10, which is a contradiction as $\gcd(10^k, \underbrace{1\cdots 11}_{N}) = 1$.

> **Remark 1.1.** This solution can also work for other choices of numbers that p must divide, such as $\underbrace{d\cdots dd}_{N}$ and a permutation of $\underbrace{d\cdots d}_{N-1}e$ for $e \neq d$. Depending on the choice of d and e, this may reduce to a finite case check of some $p \leq 7$.

§2 Solution to AMM 2026/2, proposed by Evan Chen

Let $\mathbb{R}$ denote the set of real numbers. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a strictly decreasing function such that for every real number x, the following equation holds:

$$\underbrace{f(f(\dots(f}_{2026\ f\text{'s}}(x))\dots)) = x.$$

(a) Prove that $f(f(x)) = x$ for every real number x.

(b) Prove that there exists exactly one real number x such that $f(x) = x$.

(We say a function $f\colon \mathbb{R} \to \mathbb{R}$ is *strictly decreasing* if $f(x) > f(y)$ for all real numbers $x < y$.)

In what follows, we let f^n denote the function f applied n times. For example $f^2(x)$ means $f(f(x))$.

§2.1 Part (a)

We start by noticing:

> **Claim** — The function f^2 is strictly increasing.

Proof. This follows from f being strictly decreasing: if $x < y$ then $f(y) < f(x)$ and $f^2(x) < f^2(y)$. □

Now choose any $x \in \mathbb{R}$ and consider $f^2(x)$; we claim $f^2(x) = x$. Indeed, we proceed by contradiction:

- If $x < f^2(x)$, we would then get the chain of inequalities
$$x < f^2(x) < f^4(x) < f^6(x) < \dots < f^{2026}(x) = x$$
which is a contradiction.
- Similarly $x > f^2(x)$ would give
$$x > f^2(x) > f^4(x) > f^6(x) > \dots > f^{2026}(x) = x$$
which is a contradiction.

This establishes the result.

§2.2 Another approach to part (a)

Let k be the smallest positive integer such that $f^k(x) = x$. Then $x, f(x), f^2(x), \dots, f^n(x), \dots$ is periodic with period k.

Consider the set of k numbers $S = \{x, f(x), f^2(x), \dots, f^{k-1}(x)\}$. We can sort the numbers in the set, so there exist non-negative integers $a_1, a_2, \dots, a_k$ such that $f^{a_i}(x) \in S$ and
$$f^{a_1}(x) < f^{a_2}(x) < \dots < f^{a_k}(x).$$
Since f is strictly decreasing, we have
$$f^{a_1+1}(x) > f^{a_2+1}(x) > \dots > f^{a_k+1}(x).$$

But from the periodicity and the order we recover that

$$a_1 + 1 \equiv a_k \bmod k,$$

and

$$a_k + 1 \equiv a_1 \bmod k.$$

It follows that $a_1 + 2 \equiv a_1 \bmod k$, so $2 \mid k$. Therefore $f^2(x) = x$.

§2.3 Part (b)

Since f is strictly decreasing, it certainly has *at most* one fixed point; indeed if $f(a) = a$ and $f(b) = b$ for $a < b$ would be a contradiction. Hence, we need to show that f has at least one fixed point.

To show that f has at least one fixed point, we use the following lemma:

Lemma

A decreasing surjective function must be continuous.

Proof. We can apply the definition of continuity directly: given $p \in \mathbb{R}$ and $\varepsilon > 0$, we need to establish a $\delta > 0$ such that whenever $|x - p| < \delta$, we have $|f(x) - f(p)| < \varepsilon$.

Because f is surjective, we may choose $a < p < b$ such that

$$f(p) + \varepsilon > f(a) > f(p) > f(b) > f(p) - \varepsilon.$$

Because f is strictly decreasing, the entire interval (a, b) takes values in $(f(p) - \varepsilon, f(p) + \varepsilon)$. Hence, taking $\delta = \min(|p - a|, |p - b|)$ works fine. □

To finish, define the continuous function $g \colon \mathbb{R} \to \mathbb{R}$ by

$$g(x) := f(x) - x.$$

Since f is surjective and strictly decreasing, it is easy to see $\lim_{x \to -\infty} g(x) = +\infty$ while $\lim_{x \to \infty} g(x) = -\infty$. So by the intermediate value theorem on g, it follows g must have a root, which is what we wanted to prove.

§2.4 Another approach to part (b)

We show another way to prove f has at least one fixed point without using either continuity or surjectivity. We will instead use part (a) in the form $f(f(x)) = x$, together with f being strictly decreasing.

We define the sets

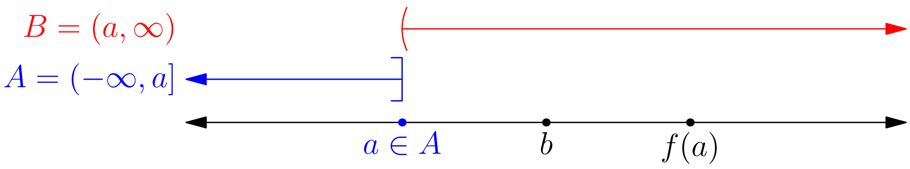
$$A := \{x \in \mathbb{R} \mid f(x) > x\}$$
$$B := \{x \in \mathbb{R} \mid f(x) < x\}.$$

Because f is strictly decreasing, the set A is downwards closed (if A contains a real number r, it contains all real numbers less than r), and the set B is upwards closed.

Assuming for contradiction that f has no fixed points, it follows A and B are in fact a partition of the real numbers. Hence, the number $a := \sup A = \inf B$ should belong to one such set. Let us assume WLOG that $a \in A$. Then it follows

$$A = (-\infty, a], \qquad B = (a, \infty).$$

An illustration is below.



Now, define

$$b := \frac{a + f(a)}{2}.$$

Then

$$a < b < f(a) \implies f(a) > f(b) > f(f(a)) = a.$$

Thus both b and $f(b)$ are larger than a, and thus belong to the set B. But these are incompatible:

$$\begin{aligned} b \in B &\implies f(b) < b \\ f(b) \in B &\implies b = f(f(b)) < f(b). \end{aligned}$$

This produces the required contradiction.

§2.5 A dyadic approach to part (b)

We establish another way to establish that f has a fixed point.

Let z_1 be a real number. Let $x_1 = \min\{z_1, f(z_1)\}$ and $y_1 = \max\{z_1, f(z_1)\}$. Then $x_1 \le y_1$, $f(x_1) = y_1$, and $f(y_1) = x_1$ (we are using that $f(f(x)) = x$ for all x). If $x_1 = y_1$ we are done, so suppose $x_1 < y_1$. Note that for any $z \in (x_1, y_1)$, then $x_1 < z < y_1$, so

$$y_1 = f(x_1) > f(z) > f(y_1) = x_1,$$

so $f(z) \in (x_1, y_1)$. Let $z_2 = \frac{x_1+y_1}{2}$. Let $x_2 = \min\{z_2, f(z_2)\}$ and $y_2 = \max\{z_2, f(z_2)\}$. As before, $x_2 \le y_2$, $f(x_2) = y_2$, and $f(y_2) = x_2$. If $x_2 = y_2$ we are done, so suppose $x_2 < y_2$. As before $z \in (x_2, y_2)$ implies $f(z) \in (x_2, y_2)$. We continue with this process of defining $z_n = \frac{x_{n-1}+y_{n-1}}{2}$, then $x_n = \min\{z_n, f(z_n)\}$ and $y_n = \max\{z_n, f(z_n)\}$ and either $x_n = y_n$ which implies $f(x_n) = x_n$ or $x_n < y_n$. Also, for any $z \in (x_n, y_n)$ we have $f(z) \in (x_n, y_n)$. Note that

$$y_n - x_n \le \frac{y_1 - x_1}{2^{n-1}}.$$

Therefore if we continue this process indefinitely (which occurs if we don't find z_n such that $f(z_n) = z_n$), we have

$$x_1 < x_2 < \cdots < x_n < y_n < y_{n-1} < \cdots < y_1.$$

Since $y_n - x_n \to 0$, by the Bolzano-Weierstrass theorem, the sequences $\{x_n\}, \{y_n\}$ converge to the same value x. But $x \in (x_n, y_n)$ for all n, so $f(x) \in (x_n, y_n)$ for all n. Therefore $f(x) = x$.

§3 Solution to AMM 2026/3, proposed by Holden Mui

(a) Show that $a! + b! + 2026$ is not a perfect square for any positive integers a and b.

(b) Prove that for every positive integer n, there exists a positive integer k such that

$$a_1! + a_2! + \cdots + a_n! + k$$

is not a perfect square for any positive integers $a_1, a_2, \ldots, a_n$.

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§3.1 Solution to (a)

Without loss of generality, assume $a \le b$.

- If $a \ge 4$, then $a! + b! + 2026 \equiv 2 \pmod 4$, so it is not square.
- If $a \in \{1, 2, 3\}$ and $b \ge 5$, then $a! + b! + 2026 \in \{2, 3\} \pmod 5$, so it is not square.
- In all other cases, $a \le b \le 4$. So $45^2 < a! + b! + 2026 < 46^2$, so it is not square.

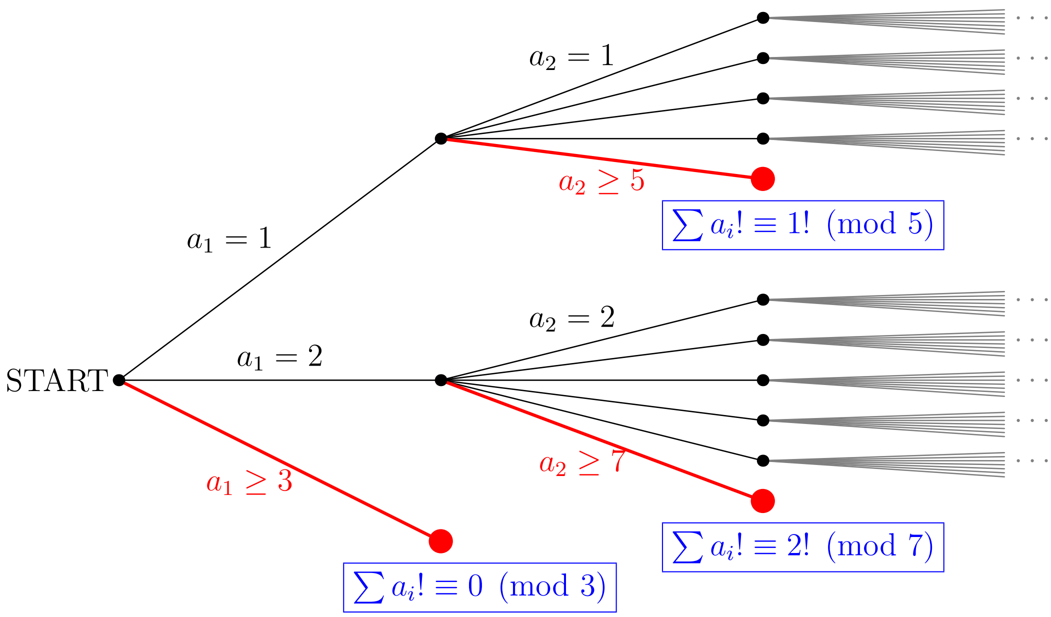
Therefore, $a! + b! + 2026$ is not square.

§3.2 Chinese remainder theorem solution to (b)

We always assume without loss of generality

$$0 < a_1 \le a_2 \le \cdots \le a_n.$$

The following branching tree with n shows our approach.



Let us write down more specific notation for our solution. Let p_1, p_2, p_3, ...be the set of odd primes in order.

Claim — We can choose a large integer $N > 0$ and residues $s_1 \pmod{p_1}$, $s_2 \pmod{p_2}$, ..., $s_N \pmod{p_N}$ such that every possible $\sum a_i!$ is in one of those residue classes.

Proof. We construct the nested tree iteratively as in the figure, depth-first, then from top to bottom. Suppose we're at a node where we have a_1, ..., $a_{\ell-1}$ are fixed and we are splitting into cases based on a_ℓ. Let p_j be the next prime not chosen. Then we create $p_j - 1$ new branches corresponding to $a_\ell = 1, 2, \dots, p_j - 1$, and one new branch for $a_\ell \ge p_j$. In that case, the value of s_j should be $a_1! + \dots + a_{\ell-1}! \pmod{p_j}$, and this works since $p_j \le a_\ell \le a_{\ell+1} \le \dots \le a_n$. □

To finish, by Chinese remainder theorem there exists an integer k such that $s_j + k \pmod{p_j}$ is not a quadratic residue for every $1 \le j \le N$. This k works by construction.

Remark. Other variants of this construction work as well; for example, one can rule out residues modulo p_j^2 instead of p_j (terminating each tree branch at $a_\ell \ge 2p_j$ so that $p_j^2 \mid a_\ell!$).

Remark. In this approach, the necessary $N = N(n)$ grows faster than exponentially in terms of n. For example, $N(1) = 3$, $N(2) = 12$, $N(3) = 203$, $N(4) = 113{,}493$, and $N(5) = 81{,}038{,}822{,}695$. It could be optimized by reusing primes since in general there are lots of quadratic nonresidues modulo any odd prime p, whereas the solution only uses one of them.

§3.3 Density solution to (b)

It suffices to show that there is some positive integer m for which

$$a_1! + a_2! + \dots + a_n! - s^2 \pmod{m!}$$

does not achieve every residue modulo $m!$. Indeed, there are m possible residues for each $a_i!$ and at most

$$m! \cdot \prod_{\text{odd prime } p \le m} \frac{(p+1)/2}{p} \le m! \left(\frac{2}{3}\right)^{\Theta(m/\ln m)}$$

possible residues for s^2 by the prime number theorem. Thus, there are

$$m^n \cdot m! \left(\frac{2}{3}\right)^{\Theta(m/\ln m)} = m! \cdot o(1)$$

possible residues modulo $m!$, as desired.

§4 Solution to AMM 2026/4, proposed by Winter Mascot

The Greek alphabet has 24 uppercase letters and 24 lowercase letters. In each cell of a 24×24 grid, we write one uppercase Greek letter and one lowercase Greek letter, with each of the 24^2 possible pairs appearing exactly once. Suppose that for each of the 48 symbols, the 24 cells containing that symbol form a connected region. Prove that the 24 cells containing the symbol Ω either all lie in the same row, or all lie in the same column.

Here, we say a set S of cells forms a *connected region* if for any two distinct cells s and s' in S, there exists a path joining them, i.e. a sequence $(s_0, s_1, \dots, s_m)$ of distinct cells in S such that $s_0 = s$, $s_m = s'$, and for each $1 \le i \le m$ the cells s_i and s_{i-1} share an edge.

In what follows, we call a set of cells forming a connected region a *polyomino*. For any Greek uppercase or lowercase letter $\oplus$, we'll use $[\oplus]$ to denote the polyomino consisting of all cells containing $\oplus$.

§4.1 Solution 1 (Author)

There are two main steps.

¶ Global step. Consider the 48 polyominoes corresponding to each symbol. Let N be the number of unordered pairs of adjacent cells belonging to the same polyomino.

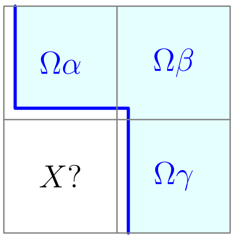
- Each of the polyominoes has area 24, so by connectivity, each of them contributes at least 23 to N. By the first condition, no pair can arise twice this way, so $N \ge 23 \cdot 48$.
- On the other hand, the grid has exactly $2 \cdot 24 \cdot 23 = 23 \cdot 48$ such pairs, so $N \le 23 \cdot 48$.

Thus $N = 23 \cdot 48$ exactly. This means that for any adjacent pair of cells, there's a polyomino that contains both of them. Moreover, any polyomino needs to contribute exactly 23 to N, which means that each polyomino must be a tree (if we treat it as a graph on 24 vertices corresponding to cells, where edges connect orthogonally adjacent cells).

¶ Local step. We prove that:

> **Claim** — The polyomino $[\Omega]$ cannot contain three cells of a 2×2 square.

Proof. Suppose otherwise. Call the three cells $\Omega\alpha$, $\Omega\beta$, $\Omega\gamma$. Then the remaining cell of the 2×2 square cannot be part of $[\Omega]$ or else $[\Omega]$ would be a tree; we refer to this cell as $X?$ below.



The neighboring pair $\{X?, \Omega\gamma\}$ doesn't appear in an uppercase-letter polyomino, so it appears in a lowercase-letter polyomino instead; thus in fact this cell should be $X\gamma$. However, the same reasoning shows it should be $X\alpha$. This produces a contradiction. $\square$

This claim easily implies $[\Omega]$ is a 1×24 or 24×1 rectangle.

§4.2 Solution 2 (Hannah Fox and Holden Mui)

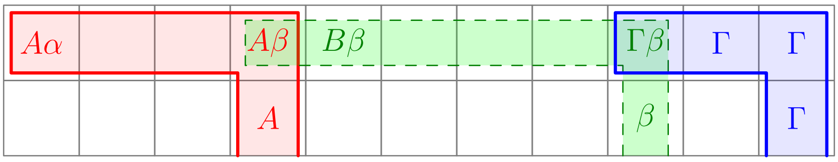
For convenience, let (i, j) denote the cell on the i-th row and j-th column of the grid, starting from the top-left corner.

We first prove the following:

> **Claim —** Either the first row of the grid all contain the same Greek letter, or the first column all contain the same Greek letter.

Proof, by Hannah Fox. Without loss of generality, let A and α be the two letters in $(1, 1)$. It is clear that among the two adjacent cells, exactly one of them contains A and the other contains α. (Otherwise there will be at least two cells labeled $A\alpha$.) That is: a polyomino cannot wrap around at a corner of the grid.

Assume without loss of generality that cell $(1, 2)$ contains A. If there are no A in the second row then we are done. Otherwise, suppose that $(2, k)$ is the leftmost such cell. Note that $k > 1$. This means that $[A]$ contains the L-shape consisting of the three cells $(1, k-1)$, $(1, k)$, and $(2, k)$. We denote the contents of $(1, k)$ as $A\beta$



It is clear that $[\beta]$ must extend to the right from that cell (or else there are at least two cells labeled $A\beta$). Moreover, $[\beta]$ must occupy at least one cell in the second row since there are less than 24 cells starting from $(1, k)$, hence forming another L-shape. We can repeat this process (alternating between uppercase and lowercase letters) until we find a polyomino containing all of $(1, 23)$, $(1, 24)$, $(2, 24)$. But we saw earlier a polyomino cannot wrap around a corner. $\square$

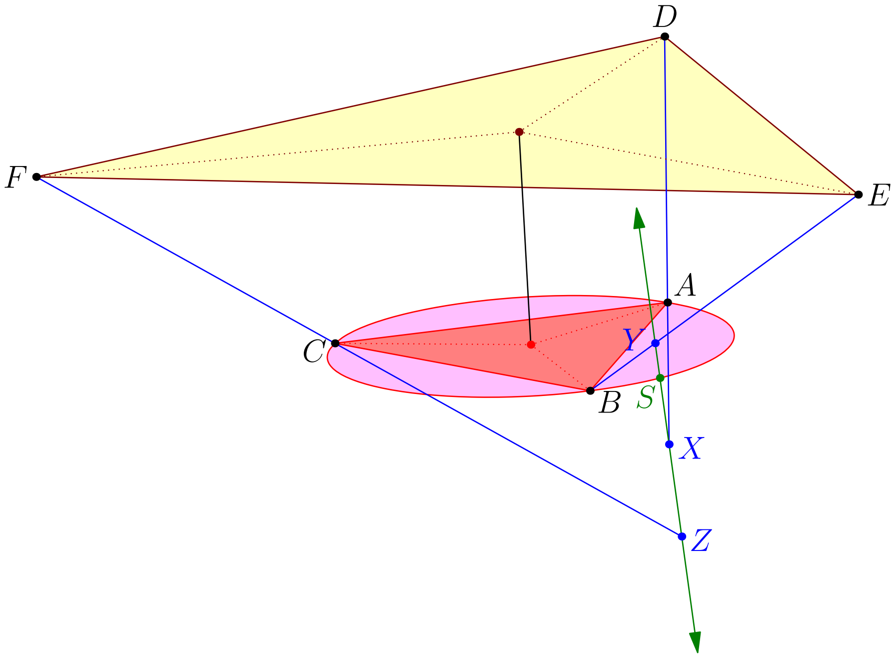
> **Remark** (Holden Mui)**.** Here is a different completion after finding the L-shape at $(1, k-1)$, $(1, k)$, and $(2, k)$. If there are no α in the second column then we are also done. Otherwise, suppose that $(\ell, 2)$ is the topmost such cell (where $\ell > 1$). By symmetry we may assume without loss of generality that $\ell \le k$.
>
> Consider the lowercase letters in cells $(1, 1), \dots, (1, k-1)$, and suppose that they are $\alpha_1 (= \alpha), \alpha_2 \dots, \alpha_{k-1}$ respectively. (Note that they must all be distinct.) Since no two lowercase polyominos overlap, we can show that cell (i, j) where $j \le k-1$ and $i + j \le k + 2$ must contain the letter α_j. In particular, cell $(\ell, 2)$ must contain the letter α_2 instead of α since $\ell + 2 \le k + 2$, which contradicts our assumption.

From the Claim above we may now assume (without loss of generality) that the first row all contains some uppercase Greek letter $\oplus$. This means that the first row must all contain different lowercase Greek letters, and we can deduce that they must remain in different columns, working from top to bottom. By the same logic we can show that all uppercase letters are in different rows. This completes the solution.

§5 Solution to AMM 2026/5, proposed by Holden Mui

In three-dimensional Euclidean space, nondegenerate tetrahedra $ABDE$, $BCEF$, and $CAFD$ are congruent to each other (in the same orientation, with vertices in this order). A line ℓ intersects each of the lines AD, BE, and CF. Prove that ℓ intersects the circumcircle of triangle ABC.

Both solutions start with the same opening steps; we will then proceed with one of two approaches.



§5.1 Common steps

Because $ABDE \cong BCEF \cong CAFD$, we know $DE = EF = FD$, so DEF is equilateral. Similarly ABC is equilateral. In what follows, let $\mathcal{P}$ be the plane of ABC.

We show that ABC and DEF are rotations around the line m through their centers:

> **Claim** — The line m through the centers of equilateral triangles ABC and DEF is perpendicular to each triangle. In particular, a $120°$ rotation about m sends AD to BE, BE to CF, and CF to AD.

Proof. Fix equilateral triangle ABC and the shape of the congruent tetrahedra. Since the lengths AD, BD, and CD are fixed, there are only two choices for point D's location, which are symmetric about plane ABC. Additionally, the possible locations for E and F are rotations of the choices for D about the line through the center of ABC perpendicular to ABC. Since DEF must be equilateral, D, E, and F must all be on the same side of ABC. □

We also verify the following degeneracy condition:

> **Claim** — The line ℓ is not parallel to $\mathcal{P}$.

Proof. Any plane parallel to $\mathcal{P}$ intersects lines AD, BE, and CF in three points forming an equilateral triangle. Since lines AD and BE are disjoint, this equilateral triangle is nondegenerate. Thus, no line in such a plane could pass through all three of these intersection points. $\square$

Hence we may define S as the unique intersection of ℓ with $\mathcal{P}$. Moreover, we denote by $X := \ell \cap \overline{AD}$, $Y := \ell \cap \overline{BE}$, $Z := \ell \cap \overline{CF}$, as promised by the problem statement.

§5.2 Solution using degree counting

Let $\mathcal{H}$ be surface formed by rotating $\overline{AD}$ around m; it contains the lines $\overline{BE}$ and $\overline{CF}$ as well. Because the tetrahedrons are nondegenerate, line m does not intersect line AD and hence does not intersect $\mathcal{H}$ either.

We will need the following fact:

Lemma

The equation for $\mathcal{H}$ is a degree-2 polynomial equation in Cartesian coordinates (x, y, z). Thus, any line not contained in $\mathcal{H}$ may intersect the surface $\mathcal{H}$ in at most two points.

Proof. Choose a coordinate system so that m is the z-axis. Parametrize line AD in the form

$$\{(x_0 + at, y_0 + bt, z_0 + ct) \mid t \in \mathbb{R}\}.$$

Because line AD is not parallel to $\mathcal{P}$, we assume $c \neq 0$.

Then $\mathcal{H}$ consists of those points which are obtainable via a rotation of this by some angle $\theta \in [0, 2\pi]$, say

$$\begin{aligned} x &= (x_0 + at)\cos\theta - (y_0 + bt)\sin\theta \\ y &= (x_0 + at)\sin\theta + (y_0 + bt)\cos\theta \\ z &= z_0 + ct. \end{aligned}$$

In particular, every such point satisfies

$$x^2 + y^2 = (x_0 + at)^2 + (y_0 + bt)^2 = \left(x_0 + a \cdot \frac{z - z_0}{c}\right)^2 + \left(y_0 + b \cdot \frac{z - z_0}{c}\right)^2 \tag{1}$$

which is a degree 2 equation, proving the first part of the lemma.

The second part follows immediately from this. Indeed, suppose we have an arbitrary line parametrized by a single real number $s \in \mathbb{R}$. Plugging it into (1) should give a polynomial of degree at most 2 in s. So either there are at most two solutions in s, or (1) holds identically for all s. $\square$

Now we bring line ℓ into the picture. Since ℓ meets $\mathcal{H}$ at X, Y, and Z, the previous lemma implies ℓ is contained inside $\mathcal{H}$. Hence S also lies on $\mathcal{H}$. But the intersection of $\mathcal{H}$ with $\mathcal{P}$ is exactly the circumcircle of triangle ABC, as desired.

> **Remark.** The surface $\mathcal{H}$ is called a *hyperboloid*; these can generally be obtained by rotating a line around a fixed axis of revolution.

§5.3 Solution using quadratics (Milan Haiman)

This solution is similar to the previous one, but it avoids discussing the surface $\mathcal{H}$ and works directly with squared distances.

Again, fix a coordinate system in (x, y, z) such that $\mathcal{P}$ is the xy-plane and m is the z-axis. For a line L not parallel to $\mathcal{P}$, we define the function $f_L \colon \mathbb{R} \to \mathbb{R}$ as follows. Parametrize L as $\{(a(t), b(t), t) \colon t \in \mathbb{R}\}$ where a and b are linear functions. Now let $f_L(t) \coloneqq a(t)^2 + b(t)^2$ be the square of the distance between L and m in the plane $z = t$. Note that f_L is a quadratic function of t, since a and b are linear.

Since AD, BE, and CF are rotations of each other about m,

$$f_{AD} = f_{BE} = f_{CF}.$$

Additionally, the points X, Y, and Z give us distinct values t_X, t_Y, and t_Z satisfying $f_{AD}(t_X) = f_\ell(t_X)$, $f_{BE}(t_Y) = f_\ell(t_Y)$, and $f_{CF}(t_Z) = f_\ell(t_Z)$. Since distinct quadratics may agree on at most 2 points, we have that $f_{AD} = f_{BE} = f_{CF} = f_\ell$. In particular, letting O be the circumcenter of triangle ABC, we have

$$OS^2 = f_\ell(0) = f_{AD}(0) = OA^2.$$

§5.4 Synthetic solution

Delete the points D, E, F from the figure; we only need to remember that lines AX, BY, and CZ are rotations around m spaced out by 120°.

Let B' be the image of B under a homothety centered at S that sends Y to X. Let C' be the image of C under a homothety centered at S that sends Z to X. Note that B' and C' are on $\mathcal{P}$. Finally, let m_A denote the line through X perpendicular to $\mathcal{P}$.

Claim — We have a spiral similarity $\triangle ABC \sim \triangle AB'C'$ centered at A.

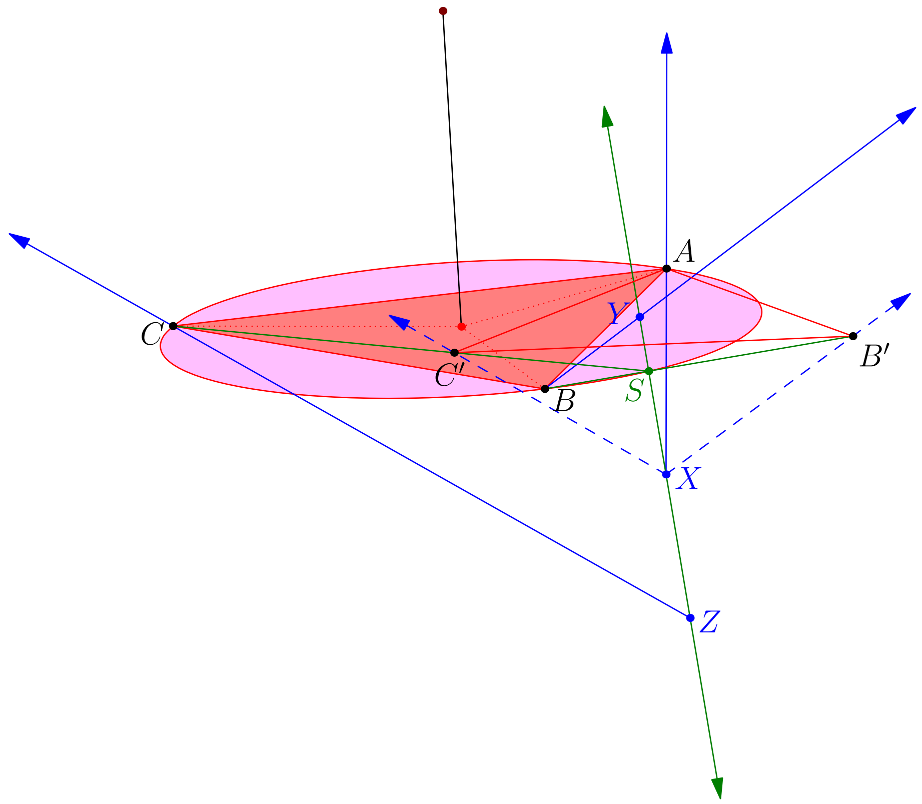
Proof. We know $\overline{BY}$ is the image of $\overline{AX}$ under a 120° rotation about m; without loss of generality, say that this rotation is clockwise. Since $\overline{B'X} \parallel \overline{BY}$, it follows $\overline{B'X}$ is also the image of $\overline{AX}$ under a 120° clockwise rotation about m_A. Similarly, $\overline{C'X}$ is the image of $\overline{AX}$ under a 120° counterclockwise rotation about m_A.

Therefore, $\triangle AB'C'$ is equilateral and has the same orientation as $\triangle ABC$. So, A is the spiral center of BC and $B'C'$. □

We also need one more nondegeneracy claim:

Claim — The points B, C, B', and C' cannot all be collinear.

Proof. Assume not. The previous claim $\triangle ABC \sim \triangle AB'C'$ then implies that in fact $B = B'$ and $C = C'$; thus $m = m_A$. By symmetry, BE and CF intersect m at the same point. This implies $ABDE$, $BCEF$, and $CAFD$ are degenerate, a contradiction. □



Hence, we have

$$S = \overline{BC} \cap \overline{B'C'}$$

but together with the spiral similarity, the problem is solved — as A is the Miquel point of quadrilateral $BCC'B'$, we deduce the needed

$$A = (SBC) \cap (SB'C') \neq S.$$

§5.5 Constructive solution

Assume A, B, and C lie in clockwise order. Let $\mathcal{Q}_A$ be the plane through m and $AD \cap \ell$. Define $\mathcal{Q}_B$ and $\mathcal{C}$ such that $\mathcal{Q}_A, \mathcal{Q}_B, \mathcal{Q}_C$ are planes containing m that form $60°$ angles with each other and $\mathcal{Q}_A$, $\mathcal{Q}_B$, and $\mathcal{Q}_C$ lie in clockwise order.

Then, the following three lines are the same:

- the reflection of AD about $\mathcal{Q}_A$
- the reflection of BE about $\mathcal{Q}_B$
- the reflection of CF about $\mathcal{Q}_C$.

Let ℓ' be this line. Then, since ℓ' contains $AD \cap \mathcal{Q}_A$, ℓ intersects AD. Similarly, ℓ' intersects BE and CF.

Assume towards a contradiction that $\ell \neq \ell'$. Then, since ℓ intersects ℓ' at $AD \cap \ell$, ℓ and ℓ' lie on a shared plane $\mathcal{P}$. Then, BE and CF must lie on this plane. However, $BCEF$ is nondegenerate, a contradiction.

Since the reflection of A over $\mathcal{Q}_A$ lies on $\ell' = \ell$ and (ABC), ℓ intersects (ABC).